JOURNAL OF APPROXIMATION THEORY **61**, 53–73 (1990)

Application of Box Splines to the Approximation of Sobolev Spaces

JAN KRZYSZTOF KOWALSKI

Polish Academy of Sciences, Institute of Mathematics, Sniadeckich 8, PL-00-950 Warsaw, Poland

Communicated by Charles K. Chui

Received March 2, 1988

1. INTRODUCTION

1.1. *Contents of the Paper*

The aim of this paper is the construction of an approximation of Sobolev spaces $W_p^m(\mathbb{R}^s)$ by some spaces of discrete functions. The approximation built here is analogous to the finite element approximation of the spaces $H^m(\mathbb{R}^s)$ presented by Aubin [1].

The following definition of an approximation of a Banach space X is used. Let us assume that H' is a set possessing an accumulation point denoted by 0 and that $H = H' \setminus \{0\}$. The system $\mathscr{A}(X, p, r) = \{(X_h, p_h, r_h)\}_{h \in H}$ is called a convergent approximation of X if X_h are Banach spaces normed by $\|\cdot\|_h$, $p_h : X_h \to X$ (prolongation) and $r_h : X \to X_h$ (restriction) are linear operators, and

$$\begin{aligned} &\exists M > 0\ \forall h \in H\ \forall f \in X \qquad \|r_h f\|_h \leqslant M \|f\|_X, \\ &\exists M > 0\ \forall h \in H\ \forall u \in X_h \qquad \|p_h u\|_X \leqslant M \|u\|_h, \end{aligned} \tag{1.1}$$

$$\forall f \in X \qquad \lim_{h \to 0} \|f - p_h r_h f\|_X = 0; \tag{1.2}$$

the approximation is called stable if

$$\exists K > 0\ \forall h \in H\ \forall u \in X_h\ \|p_h u\|_X \geqslant K \|u\|_h. \tag{1.3}$$

The construction of an approximation of spaces $W_p^m(\mathbb{R}^s)$ presented here is based on the concept of discrete and integral partitions of unity. The partition of unity is built by the use of multivariate box splines, defined by de Boor and Höllig [2]. It is observed that the functions $\pi_{(m)}$ and $\mu_{(m)}$ used by Aubin [1] for constructing the prolongation operators are box splines.

0021-9045/90 $3.00

The first section contains definitions of box splines and of the spaces of mesh functions which are used as the approximating spaces X_h. In the second section an approximation of $W_p^m(\mathbb{R}^s)$ is defined. It is shown that this approximation is bounded and convergent—that is, it satisfies conditions (1.1), (1.2)—and that the rate of convergence depends on the moduli of continuity of the given function. The necessary and sufficient conditions for the stability of the approximation are given. Last, we present an example of the application of such a method to the approximate solution of an eigenvalue differential problem.

Section 3 contains the proofs of all the results from Section 2.

1.2. *Notation*

The set of all positive real numbers is denoted by $\mathbb{R}_+$, and the set of nonnegative integers by $\mathbb{Z}_+$. A vector $x \in \mathbb{R}^s$ is written as the column $(x_1, ..., x_s)^T$. If $x, y \in \mathbb{R}^s, z \in \mathbb{R}^s_+$, then $|x|_p$ is the l_p-norm of x, $|x| = |x|_1$, $x \circ y = (x_1 y_1, ..., x_s y_s)^T$, $x/z = (x_1/z_1, ..., x_s/z_s)^T$, and if $x \in \mathbb{R}^s$ and $z \in \mathbb{R}^s_+$ or $x \in \mathbb{Z}^s_+$ and $z \in \mathbb{R}^s$ then

$$z^x = \prod_{i=1}^{s} (z_i)^{x_i}.$$

The symbol e_i denotes the unit vector of the ith axis; e is the sum of all vectors e_i.

The spaces $L_p(\mathbb{R}^s)$, $L_p(\mathbb{R}^s)^{\text{loc}}$, $W_p^m(\mathbb{R}^s)$, $W_p^m(\mathbb{R}^s)^{\text{loc}}$ $(m \in \mathbb{Z}_+, 1 \leqslant p \leqslant \infty)$ are defined as usual. The symbols $L_p(\mathbb{R}^s)_b$, $W_p^m(\mathbb{R}^s)_b$ denote the subsets of $L_p(\mathbb{R}^s)$ and $W_p^m(\mathbb{R}^s)$ consisting of all functions which vanish out of a bounded set. The norm in $L_p(\mathbb{R}^s)$ is denoted by $\|\cdot\|_p$, and the norm and seminorms in $W_p^m(\mathbb{R}^s)$ are given by

$$\|f\|_{p,m} = \left[\sum_{|k| \leqslant m} \|D^k f\|_p^p \right]^{1/p},$$

$$|f|_{p,n} = \left[\sum_{|k| = n} \|D^k f\|_p^p \right]^{1/p} \qquad (0 \leqslant n \leqslant m),$$

with the usual extension for the case $p = \infty$. The moduli of continuity are defined as follows (cf. [4]): if $f \in L_p(\mathbb{R}^s)$, $t > 0$, $r \in \mathbb{Z}_+$, then

$$\omega_r(t, f) = \sup\{ \|\Delta^r(z) f\|_p \colon |z| \leqslant t \},$$

where

$$(\Delta^r(z) f)(x) = \sum_{n=0}^{r} (-1)^{r-n} \binom{r}{n} f(x + nz), \qquad x, z \in \mathbb{R}^s;$$

if $f \in W_p^m(\mathbb{R}^s)$ then

$$\omega_r(t, \mathbb{D}^m f) = \left[\sum_{|k| = m} \omega_r(t, D^k f)^p \right]^{1/p}.$$

1.3. *Mesh*

Let us now define the mesh on $\mathbb{R}^s$ and some spaces of mesh functions. Let $H \subset \mathbb{R}^s_+$ be a bounded set of parameters with 0 as a point of density. For fixed $h \in H$, the mesh on $\mathbb{R}^s$ is the set

$$\mathbb{R}^s_h = \{x \in \mathbb{R}^s: x = l \circ h, l \in \mathbb{Z}^s\}.$$

The space of all functions $u: \mathbb{R}^s_h \to A$ (where A is a linear space) is denoted by $m(\mathbb{R}^s_h, A)$, $m(\mathbb{R}^s_h) = m(\mathbb{R}^s_h, \mathbb{R})$. The set of mesh functions vanishing out of a finite set is denoted by $m(\mathbb{R}^s_h, A)_b$. The operators of finite differences, ∂^k and ∂^k_- $(k \in \mathbb{Z}^s_+)$, are defined in the following way: if $u \in m(\mathbb{R}^s_h, A)$, $x \in \mathbb{R}^s_h$, then

$$\partial^k u(x) = h^{-k} \sum_{0 \leqslant j \leqslant k} (-e)^{k-j} \binom{k}{j} u(x + j \circ h),$$

$$\partial^k_- u(x) = h^{-k} \sum_{0 \leqslant j \leqslant k} (-e)^{k-j} \binom{k}{j} u(x - j \circ h).$$

The following spaces of mesh functions are considered:

$L_p(\mathbb{R}^s_h)$—the set of all $u \in m(\mathbb{R}^s_h)$ such that the number

$$\|u\|_p = \left[h^e \sum_{x \in \mathbb{R}^s_h} |u(x)|^p \right]^{1/p}$$

is finite; $\|\cdot\|_p$ is the norm in $L_p(\mathbb{R}^s_h)$;

$W_p^m(\mathbb{R}^s_h)$ $(m \in \mathbb{Z}_+)$—the set $L_p(\mathbb{R}^s_h)$ with the norm and seminorms

$$\|u\|_{p,m} = \left[\sum_{|k| \leqslant m} \|\partial^k u\|_p^p \right]^{1/p};$$

$$|u|_{p,n} = \left[\sum_{|k| = n} \|\partial^k u\|_p^p \right]^{1/p} \qquad (0 \leqslant n \leqslant m).$$

1.4. *Box Splines*

In this section, box spline functions, which will be used to construct an approximation of $W_p^m(\mathbb{R}^s)$, are defined. As in [3], let $X = \{x^1, ..., x^n\}$ be a set of vectors from $\mathbb{R}^s$ (not necessarily distinct), let

$$\langle X \rangle = \text{span}\, X = \mathbb{R}^s,$$

and let the number $d(X)$ be defined by

$$d(X) = \max\{m\colon \text{for all } Y \subset X,\ |Y| = m \text{ implies } \langle X \setminus Y \rangle = \mathbb{R}^s\}$$

(where $|Y|$ is the cardinality of Y). The multivariate box spline B_X is the function satisfying the identity

$$\int_{\mathbb{R}^s} B_X(x)\, f(x)\, dx = \int_{I^n} f(Xz)\, dz \qquad (\text{where } I = [0, 1]) \tag{1.4}$$

for every $f \in C(\mathbb{R}^s)$ (X is identified with the matrix $[x^1, ..., x^n]$).

We use the following matrices in our study:

$$E_k = [\underbrace{e_1, ..., e_1}_{k_1}, \underbrace{e_2, ..., e_2}_{k_2}, ..., \underbrace{e_s, ..., e_s}_{k_s}] \qquad (k \in \mathbb{Z}^s_+).$$

For the purpose of constructing an approximation of the space $W^m_p(\mathbb{R}^s)$, we introduce the following classes of matrices:

$$\mathcal{S}_m = \{X \subset \mathbb{Z}^s\colon E_{me} \subset X,\ d(X) \geqslant m\} \qquad (m \in \mathbb{Z}_+).$$

Let us observe that $\mathcal{S}_0 = \{X \subset \mathbb{Z}^s\colon \langle X \rangle = \mathbb{R}^s\}$ and if $m > 0$ then the matrix $X \setminus E_{me}$ (consisting of an arbitrary number of columns) has at least one nonzero element in each row. As an example, the functions $\mu_{(m)}$ and $\pi_{(m+1)}$, used by Aubin [1] to build an approximation of $H^m(\mathbb{R}^s)$, are box splines generated by the matrices $E_{me} \cup e$ and $E_{(m+1)e} = E_{me} \cup E_e$, respectively. These matrices belong to $\mathcal{S}_m$.

It follows from the results presented by Dahmen and Micchelli in [3] that B_X belongs to $C^{d(X)-1}(\mathbb{R}^s)$. In the same way it can be proved that

$$B_X \in W^{d(X)}_\infty(\mathbb{R}^s)_b. \tag{1.5}$$

2. Formulation of Results

In this section an approximation of $L_p(\mathbb{R}^s)$ and $W^m_p(\mathbb{R}^s)$ is constructed. In Section 2.1 the operators of restriction and prolongation which satisfy conditions (1.1) and (1.2) if $X = L_p(\mathbb{R}^s)$ are constructed. In the next section the box splines are applied to the construction of an approximation of $W^m_p(\mathbb{R}^s)$, Section 2.3 contains the conditions for stability of the approximation. Last, in Section 2.4 an example of an application of the approximation is presented.

2.1. *Approximation of* $L_p(\mathbb{R}^s)$

Let A and B be functions from $L_\infty(\mathbb{R}^s)_b$ and let us define the operators $p_h^B: m(\mathbb{R}_h^s) \to L_\infty(\mathbb{R}^s)^{\mathrm{loc}}$, $r_h^A: L_1(\mathbb{R}^s)^{\mathrm{loc}} \to m(\mathbb{R}_h^s)$ by the formulas

$$(p_h^B u)(x) = \sum_{l \in \mathbb{Z}^s} B(x/h - l)\, u(l \circ h) \tag{2.1}$$

for each $u \in m(\mathbb{R}_h^s)$ and almost every $x \in \mathbb{R}^s$;

$$(r_h^A f)(l \circ h) = h^{-e} \int_{\mathbb{R}^s} A(x/h - l)\, f(x)\, dx \tag{2.2}$$

for each $f \in L_1(\mathbb{R}^s)^{\mathrm{loc}}$, $l \in \mathbb{Z}^s$.

THEOREM 1. *Let* $A, B \in L_\infty(\mathbb{R}^s)_b$, $1 \leqslant p \leqslant \infty$. *If* $f \in L_p(\mathbb{R}^s)$ *then* $r_h^A f \in L_p(\mathbb{R}_h^s)$; *if* $u \in L_p(\mathbb{R}_h^s)$ *then* $p_h^B u \in L_p(\mathbb{R}^s)$, *and*

$$\|r_h^A f\|_p \leqslant C_1 \|f\|_p, \qquad \|p_h^B u\|_p \leqslant C_2 \|u\|_p. \tag{2.3}$$

Thus, condition (1.1) from the definition of the approximation is fulfilled. To obtain the convergence, we need more assumptions concerning the functions A, B. Let us define the sets

$$\mathscr{P}_i = \left\{ A \in L_\infty(\mathbb{R}^s)_b : \int A(x)\, dx = 1 \right\},$$

$$\mathscr{P}_d = \left\{ B \in L_\infty(\mathbb{R}^s)_b : \sum_{l \in \mathbb{Z}^s} B(x + l) = 1 \text{ for almost every } x \in \mathbb{R}^s \right\}.$$

If $A \in \mathscr{P}_i$, $B \in \mathscr{P}_d$, then $p_h^B 1 = 1$, $r_h^A 1 = 1$, and therefore the elements of $\mathscr{P}_d$ and $\mathscr{P}_i$ are called the discrete and integral partitions of unity. It is shown in Section 3.5 that

$$\mathscr{P}_d \subset \mathscr{P}_i. \tag{2.4}$$

As an example of a function from $\mathscr{P}_d$, a box spline can be taken, since (as follows from the Corollary to Proposition 3 in [2])

$$\text{if } X \in \mathscr{S}_0 \text{ then } B_X \in \mathscr{P}_d. \tag{2.5}$$

THEOREM 2. *Let* $A \in \mathscr{P}_i$, $B \in \mathscr{P}_d$. *If* $f \in L_p(\mathbb{R}^s)$, $1 \leqslant p \leqslant \infty$, *then*

$$\|f - p_h^B r_h^A f\|_p \leqslant C_3 \omega_1(|h|_\infty, f); \tag{2.6}$$

if, moreover,

$$p_h^B r_h^A g = g \qquad \forall g \in \Pi_r(\mathbb{R}^s) \tag{2.7}$$

(*where* $\Pi_r(\mathbb{R}^s)$ *is the set of all polynomials of degree not greater than* r), *then*

$$\|f - p_h^B r_h^A f\|_p \leqslant C_4 \omega_{r+1}(|h|_\infty, f). \tag{2.8}$$

It is shown in Lemma 3 that there exist functions A and B satisfying condition (2.7).

2.2. *Approximation of* $W_p^m(\mathbb{R}^s)$

The construction of the approximation of $W_p^m(\mathbb{R}^s)$ is done with box splines. If $X \in \mathscr{S}_m$ then, following properties (1.5) and (2.5), $B_X \in \mathscr{P}_d$ and $p^{B_X} u \in W_\infty^m(\mathbb{R}^s)$. The following notation will be used:

$$\text{if } X \in \mathscr{S}_0 \text{ then } p_h^X = p_h^{B_X};$$
$$\text{if } X = E_k \cup Y \text{ then } B_{k,Y} = B_X,\ p_h^{k,Y} = p_h^{B_X}. \tag{2.9}$$

Let us now formulate two lemmas which allow us to build an approximation of $W_p^m(\mathbb{R}^s)$.

LEMMA 1. *Let* $j, k \in \mathbb{Z}_+^s$, $j \leqslant k$. *If* Y *is a matrix such that* $E_{k-j} \cup Y \in \mathscr{S}_0$ *then for every* $u \in m(\mathbb{R}_h^s)$

$$D^j p_h^{k,Y} u = (-1)^{|j|} p_h^{k-j,Y} \partial_-^j u. \tag{2.10}$$

LEMMA 2. *Let* $A \in L_\infty(\mathbb{R}^s)_b$, $k \in \mathbb{Z}_+^s$, $|k| = n \leqslant m$. *Then for every function* $f \in W_1^m(\mathbb{R}^s)^{\text{loc}}$

$$\partial^k r_h^A f = r_h^{A(k)} D^k f, \qquad \partial_-^k r_h^A f = (-1)^n r_h^{A(-k)} D^k f, \tag{2.11}$$

where $A(k) = Q_e^k A$, $A(-k) = Q_{-e}^k A$, *and*

$$(Q_z^k f)(x) = \int_{I^n} f(x - z \circ E_k y)\, dy \qquad (I = [0, 1]) \tag{2.12}$$

for every $f \in L_1(\mathbb{R}^s)^{\text{loc}}$, $z \in \mathbb{R}^s$.

The following theorem is a consequence of Theorems 1, 2 and Lemmas 1, 2.

THEOREM 3. *Let* $X \in \mathscr{S}_m$, $A \in \mathscr{P}_i$. *If* $f \in W_p^m(\mathbb{R}^s)$ *then* $r_h^A f \in W_p^m(\mathbb{R}_h^s)$; *if* $u \in W_p^m(\mathbb{R}_h^s)$ *then* $p_h^X u \in W_p^m(\mathbb{R}^s)$ *and*

$$|r_h^A f|_{p,n} \leqslant C_1 |f|_{p,n}, \qquad |p_h^X u|_{p,n} \leqslant |u|_{p,n}, \tag{2.13}$$

$$|f - p_h^X r_h^A f|_{p,n} \leqslant C_5 \omega_1(|h|, \mathbb{D}^n f) \qquad (0 \leqslant n \leqslant m). \tag{2.14}$$

If, moreover, condition (2.7) *is satisfied with* $r \geqslant m$ *then*

$$|f - p_h^X r_h^A f|_{p,n} \leqslant C_6 \omega_{r+1-n}(|h|, \mathbb{D}^n f) \qquad (0 \leqslant n \leqslant m). \tag{2.15}$$

The application of box splines allows us to obtain an approximation satisfying condition (2.7). The following lemma is true.

LEMMA 3. *If* $X \in \mathscr{S}_r$, $G \in \mathscr{P}_t$, *then there exist real numbers* ξ_k ($|k| \leqslant r$) *such that if* $A(x) = \sum_{|k| \leqslant r} \xi_k G(x-k)$ *for almost every* $x \in \mathbb{R}^s$, *then condition* (2.7) *is satisfied.*

2.3. *Stability of the Approximation*

In this section the stability of the approximation is investigated. Theorem 4 gives conditions which are equivalent to stability condition (1.3). The corollary estimates the norm of $p_h^X u$ by the norms of lower degree.

THEOREM 4. *Let* $X \in \mathscr{S}_m$, $m \geqslant 0$, $1 \leqslant p \leqslant \infty$. *The following conditions are equivalent*:

$$|\det Y| \leqslant 1 \textit{ for such each } Y \subset X \textit{ such that } |Y| = s; \tag{2.16}$$

$$\textit{there exists } A \in \mathscr{P}_t \textit{ such that } r_h^A\, p_h^X u = u \textit{ for each } u \in m(\mathbb{R}_h^s); \tag{2.17}$$

$$\textit{there exists a constant } K > 0 \textit{ such that } \|p_h^X u\|_{p,m} \geqslant K\, \|u\|_{p,m} \textit{ for each } h \in H \textit{ and every } u \in W_p^m(\mathbb{R}_h^s). \tag{2.18}$$

COROLLARY. *Let* $X \in \mathscr{S}_m$, $m > 0$, $1 \leqslant p \leqslant \infty$. *If conditions* (2.16)–(2.18) *are satisfied then there exists* $C > 0$ *such that for each* $u \in W_p^m(\mathbb{R}_h^s)$,

$$\|p_h^X u\|_{p,m} \leqslant C\, |e/h|^{m-n}\, \|p_h^X u\|_{p,n} \qquad \textit{if} \quad 0 \leqslant n < m. \tag{2.19}$$

2.4. *Example of Application*

Let us consider the bilinear from, defined on the space $X = H_0^1([0, 1]^2) = \mathring{W}_2^1(I^2)$,

$$a(f, g) = \sum_{i,j=1}^{2} \langle b_{ij} D_i f, D_j g \rangle + \langle bf, g \rangle \qquad \forall f, g \in X,$$

where b_{ij}, b are sufficiently smooth functions defined on $\mathbb{R}^2$ and

$$\langle f, g \rangle = \int f(x)\, g(x)\, dx \qquad \text{for} \quad f, g \in L_2(I^2).$$

Let a be X-elliptic and let us consider the eigenvalue problem (2.20)

$$\text{find } \lambda \in \mathbb{R} \text{ and } f \in X \setminus \{0\} \text{ such that}$$

$$a(f, g) = \lambda \langle f, g \rangle \qquad \forall g \in X. \tag{2.20}$$

Let us consider a space F and a continuous linear injection $\psi: X \to F$. An external approximation of X is the system $\mathscr{A}(X, p, r, F, \psi) = \{(X_h, p_h, r_h)\}_{h \in H}$, where $p_h: X_h \to F$ and $r_h: X \to X_h$ are uniformly bounded linear operators (cf. (1.1)) and

$$\forall f \in X \qquad \lim_{h \to 0} \|\psi f - p_h r_h f\|_F = 0.$$

Let the parameter $h \in H$ be fixed and let us consider the bilinear form introduced by Aubin [1],

$$a_h(u, v) = \bar{a}(p_h u, p_h v) \qquad \forall u, v \in X_h, \tag{2.21}$$

where $\bar{a}$ is an extension of a onto the space F, that is,

$$\bar{a}(\psi f, \psi g) = a(f, g) \qquad \forall f, g \in X.$$

The approximate eigenvalue problem is stated as follows:

find $\lambda_h \in \mathbb{R}$ and $u \in X_h \setminus \{0\}$ such that

$$a_h(u, v) = \lambda_h [u. v] \qquad \forall v \in X_h \tag{2.22}$$

(where $[u, v] = h^e \sum_{l \in \mathbb{Z}^s} u(l \circ h)\, v(l \circ h)$). This problem (in a more general formulation) was considered by Regińska [5], who proved the convergence of the solutions of (2.22) to the solutions of (2.20). One of the assumptions made in [5] is that the approximation satisfies the additional condition

for every $h \in H$ there exists a subspace $V_h \subset X$ which is complementary to the null space of r_h and

$$\varepsilon_h = \sup\{\|\psi f - p_h r_h f\|_F : f \in V_h, \|f\|_X = 1\} \to 0 \qquad \text{as} \quad h \to 0. \tag{2.23}$$

Regińska [6] proved that condition (2.23) is satisfied iff

for every $h \in H$ there exists an operator $q_h: X_h \to X$ such that $r_h q_h u = u$ for each $u \in X_h$ and

$$\varepsilon'_h = \sup\{\|p_h u - \psi q_h u\|_F : u \in X_h, \|u\|_h = 1\} \to 0 \qquad \text{as} \quad h \to 0. \tag{2.24}$$

Now, let us build an approximation of X. As in [1] and [6], let $F = L_2(I^2) \times L_2(I^2, D_1) \times L_2(I^2, D_2)$, where $L_2(\Omega, D_i)$ is the space $\{f \in L_2(\Omega): D_i f \in L_2(\Omega)\}$, normed by $\|f\|_{(i)}^2 = \|f\|_2^2 + \|D_i f\|_2^2$, and let $\psi f = (f, f, f)$ for $f \in X$.

Let $H = \{(1/n, 1/n)\}_{n=2}^{\infty}$; we will write h instead of (h, h). Let X, Y, Z be matrices from $\mathscr{S}_0$ such that $B_Y \in L_\infty(\mathbb{R}^2, D_1)$. $B_Z \in L_\infty(\mathbb{R}^2, D_2)$; let

$N = \operatorname{supp} B_X \cup \operatorname{supp} B_Y \cup \operatorname{supp} B_Z$, and $I_h^2 = \{lh\colon l \in \mathbb{Z}^2, lh + Nh \subset I^2\}$. Let $X_h = \{u \in m(\mathbb{R}_h^2)\colon u(lh) = 0 \text{ if } l \notin I_h^2\}$, and

$$pu = (p^X u, p^Y u, p^Z u)|_{I^2}, \tag{2.25}$$

where p^X, p^Y, p^Z are the prolongation operators defined by (2.9) and (2.1) (the subscript "$_h$" is suppressed). Further, let A be a function from $\mathscr{P}_i$, and let

$$(rg)(lh) = (r^A \tilde{g})(lh) \text{ if } lh \in I_h^2, \qquad (rg)(lh) = 0 \text{ if } lh \notin I_h^2, \tag{2.26}$$

where $\tilde{g}$ is the function from $W_2^1(\mathbb{R}^2)$ defined by

$$\tilde{g}(x) = g(x) \text{ if } x \in I^2, \qquad \tilde{g}(x) = 0 \text{ if } x \notin I^2.$$

It follows from the results presented in Sections 2.1 and 2.2 and from the results of Aubin [1] concerning the approximation on subsets of $\mathbb{R}^2$ that $\mathscr{A}(X, p, r, F, \psi)$ is an external approximation of X.

Let us investigate the approximations generated by different matrices X, Y, Z. Regińska proved in [6] that the choice proposed by Aubin [1], $X = [e_1, e_2]$, $Y = [e_1, e_1, e_2]$, $Z = [e_1, e_2, e_2]$, does not satisfy condition (2.24). However, the following result is true.

LEMMA 4. *Let us assume that $X \in \mathscr{S}_0$ and that Y belongs to $\mathscr{S}_1$ and satisfies condition* (2.16). *Let $Z = Y$. Then there exists a function $A \in \mathscr{P}_i$ such that the approximation* (2.25)–(2.26) *fulfills* (2.24).

Let us now construct the approximate problem. Let the approximate bilinear form be defined by (2.21) with

$$\bar{a}(\bar{f}, \bar{g}) = \sum_{i,j=1}^{2} \langle b_{ij} D_i f_i, D_j g_j \rangle + \langle b f_0, g_0 \rangle \tag{2.27}$$

if $\bar{f} = (f_0, f_1, f_2)$, $\bar{g} = (g_0, g_1, g_2) \in F$. It is proved in Section 3.7 that

$$a_h(u, v) = -\sum_{i,j=1}^{2} [\partial_j r^{Y(j)} b_{ij} p^{Y(i)} \partial_i^- u + r^X b p^X u, v], \tag{2.28}$$

where $Y(i) = Y \setminus e_i$, $\partial_i = \partial^{e_i}$, $\partial_i^- = -\partial_-^{e_i}$. Thus, if $Y = [e_1, e_2, e]$, $X = [e_1, e_2]$, then we obtain the difference scheme

$$-\sum_{i=1}^{2} [\partial_i \beta_{ii} \partial_i^- + \partial_{3-i}(t_i \beta_{i,3-i}^i)\, \partial_i + \partial_i^- \beta_{3-i,i}^i \partial_{3-i}^-]\, u + \beta_0 u = \lambda_h u,$$

where

$$\beta_{ii}(lh)=\int_{A_i} b_{ii}(lh+\xi h)\,d\xi,$$

$$A_i=\{(x_1,x_2): 0<x_i<1,\ x_i<x_{3-i}<1+x_i\},$$

$$\beta_{ij}^k(lh)=\int_{B_k} b_{ij}(lh+\xi h)\,d\xi,$$

$$B_k=\{(x_1,x_2): 0<x_{3-k}<x_k<1\},$$

$$\beta_0(lh)=\int_{I^2} b_0(lh+\xi h)\,d\xi,$$

$$t_i u(lh)=u(lh+e_i h)$$

($i, j, k=1, 2, l\in\mathbb{Z}^2$). This scheme is based on seven points; for any other choice of Y the scheme connects more points.

3. Proofs

In this section all the results are proved. First, some definitions are introduced which will be used in the proof. Section 3.2 contains some auxiliary formulas, and in the next sections all the lemmas and theorems are proved.

3.1. *Definitions*

First, if f, g are measurable functions on $\mathbb{R}^s$ then

$$\langle f, g\rangle=\int_{\mathbb{R}^s} f(x)\,g(x)\,dx$$

(if the integral is well defined).

The operators of translation, T_z, difference, S_z^k, and multiplication of the argument, M_y ($z\in\mathbb{R}^s$, $y\in\mathbb{R}^s_+$, $k\in\mathbb{Z}^s_+$), are defined as follows. If $f\in L_1(\mathbb{R}^s)^{\mathrm{loc}}$, $x\in\mathbb{R}^s$, then

$$(T_z f)(x)=f(x+z),\qquad (M_y f)(x)=f(x\circ y),$$
$$S_z^k f=\sum_{0\leqslant j\leqslant k}(-e)^{k-j}\binom{k}{j}T_{j\circ z}f. \tag{3.1}$$

The following power functions are used:

$$c_k(x)=x^k/k! \qquad \text{for} \quad x\in\mathbb{R}^s,\ k\in\mathbb{Z}^s_+.$$

If X is a linear topological space, $u \in m(\mathbb{R}_h^s)$, $v \in m(\mathbb{R}_h^s, X)$, then

$$[u, v] = h^e \sum_{l \in \mathbb{Z}^s} u(l \circ h)\, v(l \circ h),$$

provided that the series is convergent in the topology of X.

Let $A \in L_\infty(\mathbb{R}^s)_b$. We define the function $v_h^A : \mathbb{R}_h^s \to L_\infty(\mathbb{R}^s)_b$ by

$$v_h^A(l \circ h) = h^{-e} T_{-l \circ h} M_{e,h} A \text{ for each } l \in \mathbb{Z}^s,$$

$$\text{that is, } (v_h^A(l \circ h))(x) = h^{-e} A(x/h - l). \tag{3.2}$$

Using these definitions, we can represent r_h^A and p_h^B in the form

$$(r_h^A f)(l \circ h) = \langle v_h^A(l \circ h), f \rangle \qquad \text{or} \qquad r_h^A f = \langle v_h^A, f \rangle; \tag{3.3}$$

$$p_h^B u = [u, v_h^B]. \tag{3.4}$$

Similarly to definition (2.9), if $X \in \mathcal{S}_0$ then we write v_h^X instead of $v_h^{B_X}$; if $X = E_k \cup Y$ then $v_h^{k,Y} = v_h^{B_X}$.

Finally, let $A, B \in L_\infty(\mathbb{R}^s)_b$ and let us define the function W_{AB} by the formula

$$W_{AB}(x, z) = \sum_{l \in \mathbb{Z}^s} A(x + z - l)\, B(x - l), \qquad x, z \in \mathbb{R}^s. \tag{3.5}$$

3.2. *Auxiliary Formulas*

Let us start from the following properties of box splines, which follow from formulas (2.7) and (2.10) in [2]:

$$D^j B_{k,Y} = (-1)^{|j|} S_{-e}^j B_{k-j,Y} \qquad \text{if } j, k \in \mathbb{Z}_+^s,\ j \leqslant k,\ E_{k-j} \cup Y \in \mathcal{S}_0; \tag{3.6}$$

$$B_{j+k,Y} = Q_e^j B_{k,Y} \qquad \text{if } j, k \in \mathbb{Z}_+^s,\ E_k \cup Y \in \mathcal{S}_0. \tag{3.7}$$

The following property of the moduli of continuity, which is taken from [4], will also be used in the proof: there exist positive constants M_1, M_2 (depending on p, r) such that for every function $f \in L_p(\mathbb{R}^s)$

$$M_1 \omega_r(t, f) \leqslant K_r(t^r, f) \leqslant M_2 \omega_r(t, f), \tag{3.8}$$

where $K_r(t, f) = \inf\{ \|f - g\|_p + t\, |g|_{p,r} : g \in H_p^r(\mathbb{R}^s)\}$, and $H_p^r(\mathbb{R}^s) = W_p^r(\mathbb{R}^s)$ if $p < \infty$, $H_\infty^r(\mathbb{R}^s) = C^r(\mathbb{R}^s)$.

Now, let us give some formulas which can be proved easily. If $f \in L_1(\mathbb{R}^s)^{\mathrm{loc}}$ and $g \in L_\infty(\mathbb{R}^s)_b$ or $f \in L_p(\mathbb{R}^s)$ and $g \in L_{p'}(\mathbb{R}^s)$, $1/p + 1/p' = 1$, then

$$\langle S_z^k f, g \rangle = \langle f, S_{-z}^k g \rangle, \qquad \langle Q_z^k f, g \rangle = \langle f, Q_{-z}^k g \rangle \tag{3.9}$$

(the operator Q_z^k is defined by (2.12)).

If $f \in L_1(\mathbb{R}^s)^{\text{loc}}$ then

$$T_z M_y f = M_y T_{z \circ y} f, \qquad Q_z^k M_y f = M_y Q_{y \circ z}^k f, \qquad T_z Q_y^k f = Q_y^k T_z f; \quad (3.10)$$

if $f \in W_1^m(\mathbb{R}^s)^{\text{loc}}$, $|k| \leqslant m$, then

$$D^k M_y f = y^k M_y D^k f, \qquad D^k T_z f = T_z D^k f, \qquad S_y^k f = y^k Q_{-y}^k D^k f. \quad (3.11)$$

Next,

$$c_k(x+y) = \sum_{0 \leqslant j \leqslant k} c_{k-j}(x)\, c_j(y), \qquad c_k(x \circ y) = x^k c_k(y); \quad (3.12)$$

$$D^j c_k = c_{k-j} \text{ if } 0 \leqslant j \leqslant k, \qquad D^j c_k = 0 \text{ for other vectors } j \in \mathbb{Z}_+^s. \quad (3.13)$$

If $u \in m(\mathbb{R}_h^s)$ and $v \in m(\mathbb{R}_h^s, X)_b$ or $u \in L_p(\mathbb{R}_h^s)$ and $v \in L_{p'}(\mathbb{R}_h^s)$, then

$$[u, \partial^k v] = [\partial_-^k u, v]. \quad (3.14)$$

If $A \in L_\infty(\mathbb{R}^s)_b$, $k \in \mathbb{Z}_+^s$, then

$$S_h^k v_h^A = h^k \partial_-^k v_h^A, \qquad S_{-h}^k v_h^A = h^k \partial^k v_h^A \quad (3.15)$$

(that is, for each $x \in \mathbb{R}^s$, $S_h^k(v_h^A(x)) = h^k(\partial_-^k v_h^A)(x)$).

Now, let us give several formulas which will be proved in the next section. If $A, B \in L_\infty(\mathbb{R}^s)_b$, $k \in \mathbb{Z}_+^s$, $u \in m(\mathbb{R}_h^s)$, $f \in L_1(\mathbb{R}^s)^{\text{loc}}$, then

$$S_h^k p_h^B u = h^k p_h^B \partial^k u, \quad (3.16)$$

$$\partial^k r_h^A f = h^{-k} r_h^A S_h^k f, \qquad \partial_-^k r_h^A f = h^{-k} r_h^A S_{-h}^k f. \quad (3.17)$$

If $j, k \in \mathbb{Z}_+^s$, $E_k \cup Y \in \mathscr{S}_0$, $u \in m(\mathbb{R}_h^s)$, then

$$p_h^{j+k, Y} u = Q_h^j(p_h^{k, Y} u). \quad (3.18)$$

If $A \in L_\infty(\mathbb{R}^s)_b$, $k \in \mathbb{Z}_+^s$, then

$$r_h^A c_k = \sum_{0 \leqslant j \leqslant k} \alpha_j^A h^j c_{k-j}, \qquad \alpha_j^A = \langle A, c_j \rangle; \quad (3.19)$$

if $X \in \mathscr{S}_m$, $k \in \mathbb{Z}_+^s$, $|k| \leqslant m$ then

$$p_h^X c_k = \sum_{0 \leqslant j \leqslant k} \beta_j^X h^j c_{k-j}, \qquad \beta_j^X = \sum_{l \in \mathbb{Z}^s} B_X(-l)\, c_j(l). \quad (3.20)$$

3.3. *Proofs of Auxiliary Formulas*

Below, the parameter h is fixed, and hence the subscript „$_h$" is suppressed.

Proof of Formula (3.16). Following definition (3.1) and formulas (3.4), (3.15), (3.14), we have

$$\begin{aligned} S_h^k p^B u &= [S_h^k v^B, u] = h^k[\partial_-^k v^B, u] \\ &= h^k[v^B, \partial^k u] = h^k p^B \partial^k u. \quad ▮ \end{aligned}$$

Proof of (3.17). According to (3.3), (3.15), (3.9), we have

$$\begin{aligned} \partial^k r^A f &= \langle \partial^k v^A, f \rangle = h^{-k} \langle S_{-h}^k v^A, f \rangle \\ &= h^{-k} \langle v^A, S_h^k f \rangle = h^{-k} r^A S_h^k f; \end{aligned}$$

the second formula can be proved similarly. ▮

Proof of (2.10) *and* (3.18). We prove here that formula (2.10) from Lemma 1 is true, since it will be used in the next proofs. It follows from (3.2), (3.11), (3.6), (3.10), and (3.15) that

$$\begin{aligned} D^j v^{k,Y}(l \circ h) &= h^{-e} D^j T_{-l \circ h} M_{e h} B_{k,Y} \\ &= h^{-e-j} T_{-l \circ h} M_{e h} D^j B_{k,Y} \\ &= h^{-e-j}(-1)^{|j|}\, T_{-l \circ h} M_{e \cdot h} S_{-e}^j B_{k-j,Y} \\ &= h^{-e-j}(-1)^{|j|}\, S_{-h}^j T_{-l \cdot h} M_{e h} B_{k-j,Y} \\ &= (-1)^{|j|}\, h^{-j} S_{-h}^j v^{k-j,Y} \\ &= (-1)^{|j|}\, \partial^j v^{k-j,Y}. \end{aligned}$$

Therefore, due to (3.4) and (3.14),

$$D^j p^{k,Y} u = (-1)^{|j|}[\partial^j v^{k,Y}, u] = (-1)^{|j|}[v^{k,Y}, \partial_-^j u],$$

and formula (2.10) is proved. Formula (3.18) can be obtained similarly by use of (3.2), (3.7), (3.10), and (3.4). ▮

Proof of Formula (3.19). Introducing a new variable of integration, $y = x/h - l$, into definition (2.2), and using (3.12), we obtain

$$\begin{aligned} r^A c_k(l \circ h) &= \int_{\mathbb{R}^s} A(y)\, c_k(y \circ h + l \circ h)\, dy \\ &= \sum_{0 \leqslant j \leqslant k} c_{k-j}(l \circ h) \int_{\mathbb{R}^s} A(y)\, h^j c_j(y)\, dy, \end{aligned}$$

which was to be proved. ▮

Proof of Formula (3.20). The proof is carried out by induction with respect to m. If $m = 0$ then (3.20) holds; let us thus assume that it also is true for some $m = n-1 \geqslant 0$. Let $X \in \mathcal{S}_n$ and $|k| \leqslant n$; let us represent X in the form $X = E_{ne} \cup Y$. Let $1 \leqslant i \leqslant s$ and $z = e_i$. According to (2.10), we have

$$D^z p^{ne, Y} c_k = -p^{ne-z, Y} \partial_-^z c_k. \tag{3.21}$$

Let us denote k_i by d; using (3.12) and the formula

$$c_k(e_i) = 1/r! \text{ if } k = re_i,\ r \in \mathbb{Z}_+, \qquad c_k(e_i) = 0 \text{ for other } k,$$

we deduce that

$$\partial_-^z c_k = -\sum_{r=0}^{d-1} \frac{(-h)^{rz}}{(r+1)!} c_{k-z-rz}.$$

The matrix $E_{ne-z} \cup Y$ belongs to $\mathcal{S}_m$, hence, applying the inductive assumption to the right-hand side of (3.21), we obtain

$$D^z p^{ne, Y} c_k = \sum_{r=0}^{d-1} \frac{(-h)^{rz}}{(r+1)!} \sum_{0 \leqslant j \leqslant k-z-rz} \beta_j^{ne-z, Y} h^j c_{k-z-rz-j}.$$

Changing the variables of summation $(q = j + rz)$, we come to the formula

$$\begin{gathered} D^z p^{ne, Y} c_k = \sum_{0 \leqslant q \leqslant k-z} \gamma_{qz}^{ne, Y} h^q c_{k-z-q}, \\ \gamma_{qz}^{ne, Y} = \sum_{r=0}^{q_i} \frac{(-1)^r}{(r+1)!} \beta_{q-rz}^{ne-z, Y}. \end{gathered} \tag{3.22}$$

Formula (3.22) is true for each vector $z = e_i$, hence $p^{ne, Y} c_k$ is a polynomial of the form

$$p^{ne, Y} c_k = \sum_{0 \leqslant q \leqslant k} \varepsilon_{qk}^{ne, Y}(h)\, c_{k-q}. \tag{3.23}$$

Computing $D^z p^{ne, Y} c_k$ from (3.23) and (3.13), and comparing the result with (3.22), we obtain the formula $\varepsilon_{qk}^{ne, Y}(h) = h^{ne} \gamma_{qz}^{ne, Y}$, from which we deduce that $\varepsilon_{qk}^{ne, Y}(h)$ does not depend on k. Thus,

$$p^X c_k = \sum_{0 \leqslant j \leqslant k} \beta_j^X h^j c_{k-j} \qquad \text{if} \quad X \in \mathcal{S}_{m+1},\ |k| \leqslant m+1, \tag{3.24}$$

and β_q^X can be calculated by taking $k = q$ and evaluating both sides of (3.24) at 0 with formulas (2.1) and (3.12). ∎

3.4. *Approximation of* $L_p(\mathbb{R}^s)$

In this section the results from Section 2.1 are proved. All the proofs are carried out in the case $p < \infty$; the extension for the case $p = \infty$ is simple.

As in the previous section, the parameter h and the functions A, B are fixed, and hence we write p, r instead of p_h^B, r_h^A.

Proof of Theorem 1. Following the definitions of rf and of the norm, we have

$$\|rf\|_p \leqslant \left[h^{e(1-p)} \sum_{l \in \mathbb{Z}^s} \left(\int |A(x/h-l)|\, |f(x)|\, dx \right)^p \right]^{1/p}.$$

Applying Hölder's inequality to the product of $|A(x/h-l)|^{1/p}\, |f(x)|$ and $|A(x/h-l)|^{1/p'}$, we obtain estimate (2.3) with

$$C_1 = \|A\|_1^{1/p'}\, \|\Sigma_A\|_\infty^{1/p}, \qquad \text{where} \quad \Sigma_F(x) = \sum_{l \in \mathbb{Z}^s} |F(x+l)|.$$

The estimate for $\|pu\|_p$ can be obtained similarly, and

$$C_2 = \|B\|_1^{1/p}\, \|\Sigma_B\|_\infty^{1/p'}. \quad ▮$$

Proof of Formula (2.4). If $A \in \mathscr{P}_d$ then

$$\int_{\mathbb{R}^s} A(x)\, dx = \sum_{l \in \mathbb{Z}^s} \int_{l+I^s} A(x)\, dx = \sum_{l \in \mathbb{Z}^s} \int_{I^s} A(x+l)\, dx$$
$$= \int_{I^s} \sum_{l \in \mathbb{Z}^s} A(x+l)\, dx = 1$$

(the sum commutes with the integral, since supp A is bounded), and thus $A \in \mathscr{P}_t$, which was to be shown. ▮

Proof of Theorem 2. The proof consists of three parts. First, we prove formula (2.6). Next, we prove Lemma 5, which gives the estimate of $\|f - prf\|_p$ in the case where $f \in W_p^{r+1}(\mathbb{R}^s)$ and (2.7) is fulfilled. The third part is the proof of inequality (2.8).

Proof of Inequality (2.6). Following definitions (2.1), (2.2), and (3.5) we have

$$prf(x) = \int_\Omega W_{AB}(x/h, z)\, f(x + z \circ h)\, dz, \tag{3.25}$$

where $\Omega = \text{supp}\, B - \text{supp}\, A$. It can be shown that supp $W_{AB}(x, \cdot) \subset \Omega$ for almost every $x \in \mathbb{R}^s$, and

$$\|W_{AB}\|_\infty \leqslant M_3 = \min(\|B\|_\infty\, \|\Sigma_A\|_\infty,\ \|A\|_\infty\, \|\Sigma_B\|_\infty). \tag{3.26}$$

Moreover, since $B \in \mathscr{P}_d$ and $A \in \mathscr{P}_i$, for each $x \in \mathbb{R}^s$,

$$\int_\Omega W_{AB}(x, z)\, dz = 1.$$

Therefore,

$$\|f - prf\|_p^p \leqslant \int_{\mathbb{R}^s} \left[\int_\Omega W_{AB}(x/h, z)\, S_{z \circ h} f(x)\, dz \right]^p dx, \tag{3.27}$$

and hence

$$\|f - prf\|_p^p \leqslant M_3^p \int_{\mathbb{R}^s} \left[\int_\Omega |S_{z \circ h} f(x)|\, dz \right]^p dx.$$

Applying Hölder's inequality and changing the order of integration, we obtain the estimate

$$\|f - prf\|_p^p \leqslant M_3^p \operatorname{vol}(\Omega)^{p-1} \int_\Omega \|S_{z \circ h} f\|_p^p\, dz.$$

Hence, formula (2.6) is satisfied by

$$C_3 = M_3 \operatorname{vol}(\Omega)(1 + \operatorname{diam}(\Omega \cup \{0\})). \quad \blacksquare$$

LEMMA 5. *If the assumptions of Theorem 2 are satisfied and* $f \in W_p^{r+1}(\mathbb{R}^s)$ *then*

$$\|f - prf\|_p \leqslant M_4\, |h|_\infty^{r+1}\, |f|_{p,r+1}. \tag{3.28}$$

Proof. First, if condition (2.7) is satisfied then, following formulas (3.25) and (3.12), for every $k \in \mathbb{Z}_+^s$ such that $|k| \leqslant r$ we have

$$\begin{aligned} c_k(x) &= \int_\Omega W_{AB}(x/h, z)\, c_k(x + z \circ h) dz \\ &= \sum_{0 \leqslant j \leqslant k} h^j \int_\Omega W_{AB}(x/h, z)\, c_j(z)\, dz\, c_{k-j}(x). \end{aligned}$$

Hence,

$$\int_\Omega W_{AB}(x/h, z)\, c_j(z)\, dz = \delta_{j0} \qquad \text{if} \quad j \in \mathbb{Z}_+^s,\ |j| \leqslant r. \tag{3.29}$$

Now, let $f \in W_p^{r+1}(\mathbb{R}^s)$. Substituting Taylor's formula

$$S_{z:h} f(x) = \sum_{0<|k|\leqslant r} h^k c_k(z)\, D^k f(x) + \sum_{|k|=r+1} h^k c_k(z) \int_I (r+1)(1-\xi)^r D^k f(x+\xi z \circ h)\, d\xi$$

into (3.27) and using formula (3.29), we obtain

$$\|f - prf\|_p^p = \int_{\mathbb{R}^s} \left\{ \int_\Omega W_{AB}(x/h, z) \sum_{|k|=r+1} h^k c_k(z) \times \int_I (r+1)(1-\xi)^r D^k f(x + \xi z \circ h)\, d\xi\, dz \right\}^p dx.$$

By applying estimate (3.26) and Hölder's inequality, and changing the order of integration, we can deduce that (3.28) holds. ∎

Proof of Formula (2.8). Let us take an arbitrary function $g \in H_p^{r+1}(\mathbb{R}^s)$. Then

$$\|f - prf\|_p \leqslant \|f-g\|_p + \|g - prg\|_p + \|pr(g-f)\|_p.$$

Applying Theorem 1 and Lemma 5, we obtain

$$\|f - prf\|_p \leqslant (1 + C_1 C_2)\, \|f-g\|_p + M_4\, |h|^{r+1}\, |g|_{p,r+1}.$$

We can derive formula (2.8) from this estimate using inequality (3.8). ∎

3.5. *Approximation of* $W_p^m(\mathbb{R}^s)$

Lemma 1 was proved in Section 3.3. Let us prove the remaining results of Section 2.2. As in Section 3.3, the subscripts "$_h$" are suppressed.

Proof of Lemma 2. According to (3.11), $S_h^k f = h^k Q_{-h}^k D^k f$. Thus, it follows from (3.17) and (3.9) that $\partial^k r^A f = \langle Q_h^k v^A, D^k f \rangle$. But, following (3.2) and (3.10),

$$\begin{aligned} Q_h^k v^A(l \circ h) &= h^{-e} Q_h^k T_{-l \circ h} M_{e\,h} A \\ &= h^{-e} T_{-l \circ h} M_{e,h} Q_e^k A = v^{A(k)}(l \circ h). \end{aligned}$$

Hence, the first formula from (2.11) is proved; the second one can be obtained analogously. ∎

Proof of Theorem 3. If $X \in \mathscr{S}_m$ then $X = E_{me} \cup Y$ and $E_{me-l} \cup Y \in \mathscr{S}_0$ if

$j \in \mathbb{Z}^s_+$, $|j| \leqslant m$. Moreover, if $A \in \mathscr{P}_i$ then $A(j) \in \mathscr{P}_i$. Hence, it follows from Lemmas 1 and 2 that

$$D^j p^X r^A f = p^{me-j, Y} r^{A(j)} D^j f, \tag{3.30}$$

and inequalities (2.13), (2.14) can be deduced from Theorems 1 and 2 (the constant in the second inequality in (2.13) equals 1, since B_X is nonnegative and belongs to $\mathscr{P}_d$). If condition (2.7) is satisfied, then putting $f = c_k$ ($|k| \leqslant r$) into (3.30) and applying formula (3.13), we obtain the equality

$$p^{me-j, Y} r^{A(j)} c_q = c_q \qquad \text{if} \quad |q| \leqslant m - r,$$

which is the analogue of (2.7) for $B_{me-j, Y}$ and $A(j)$. Thus, estimate (2.15) also is satisfied. ∎

Proof of Lemma 3. It follows from definitions (2.1), (2.2) that if A is defined as in the lemma, then for every $f \in L_1(\mathbb{R}^s)^{\text{loc}}$,

$$p^X r^A f = \sum_{|k| \leqslant r} \xi_k T_{k \circ h} p^X r^G f.$$

Hence, taking $f = c_j$ where $|j| \leqslant r$, and applying (3.19), (3.20) and (3.12), we obtain the equality

$$p^X r^A c_j = \sum_{|k| \leqslant r} \xi_k \sum_{0 \leqslant q \leqslant j} \sum_{0 \leqslant n \leqslant q} \sum_{0 \leqslant i \leqslant n} \alpha^G_{j-q} \beta^X_{q-n} c_{n-i}(k) h^{j-i} c_i,$$

which in an elementary way can be transformed to

$$p^X r^A c_j = \sum_{0 \leqslant i \leqslant j} \sum_{|k| \leqslant r} \xi_k \sum_{0 \leqslant q \leqslant i} \sum_{0 \leqslant n \leqslant q} \alpha^G_{i-q} \beta^X_{q-n} c_n(k) h^i c_{j-i}.$$

Hence, the system of equations, $p^X r^A c_j = c_j$ if $|j| \leqslant r$, is equivalent to the system of linear equations

$$\sum_{|k| \leqslant r} \xi_k \sum_{0 \leqslant q \leqslant i} \sum_{0 \leqslant n \leqslant q} \alpha^G_{i-q} \beta^X_{q-n} c_n(k) = \delta_{0i}, \qquad |i| \leqslant r,$$

which is uniquely solvable due to the fact that the matrix $(c_n(k))_{|n| \leqslant r, |k| \leqslant r}$ is nonsingular and $\alpha^G_0 = \beta^X_0 = 1$. Hence, the lemma is proved. ∎

3.6. *Stability of the Approximation*

Proof of Theorem 4. First, we prove that (2.16) implies (2.17). It follows from the definitions of p^X and r^A that condition (2.17) is equivalent to

$$\forall l \in \mathbb{Z}^s \ \langle A, T_l B_X \rangle = \delta_{0l}. \tag{3.31}$$

Let N be an arbitrary open set such that $N \cap \operatorname{supp} B_X \neq \emptyset$ and let $K = \{l \in \mathbb{Z}^s : N \cap (\operatorname{supp} B_X - l) \neq \emptyset\}$. If (2.16) holds then, according to the theorem of Dahmen and Micchelli [3], the set $\{(T_l B_X)|_N : l \in K\}$ is linearly independent. We want to find a function A which satisfies (3.31) and the conditions

$$\operatorname{supp} A \subset N, \qquad A|_N = \sum_{v \in K} \gamma_v G_v, \qquad \text{where} \quad G_v = (T_v B_X)|_N.$$

Substituting these formulas into (3.31) we obtain the system of linear equations for γ_v

$$\sum_{v \in K} \gamma_v \langle G_v, G_l \rangle = \delta_{0l}, \qquad l \in K,$$

which is uniquely solvable due to the linear independence of the functions G_v. Summing up Eqs. (3.31) with respect to l, we can show that $A \in \mathscr{P}_d$; therefore condition (2.17) is satisfied.

Now, let us prove that (2.17) implies (2.18). Following Theorem 3, if (2.17) holds then for every $u \in W_p^m(\mathbb{R}_h^s)$,

$$\|u\|_{p,m} = \|r^A p^X u\|_{p,m} \leqslant C_1 \|p^X u\|_{p,m}.$$

Thus, (2.18) is satisfied.

Finally, we must prove that if condition (2.16) is not satisfied then (2.18) does not hold. Let Y be such a submatrix of X that $v = |\det Y| > 1$. It follows from definition (1.4) that $B_Y = v^{-1} \chi_{N_Y}$, where $N_Y = \operatorname{int}(\operatorname{supp} B_Y)$. Further, since B_Y belongs to $\mathscr{P}_d$, the set $K_Y = \{l \in \mathbb{Z}^s : N_Y \cap (N_Y - l) \neq \emptyset\}$ contains v elements and every vector $i \in \mathbb{Z}^s$ can be uniquely represented in the form $i = l + Yj$, where $l \in K_Y$, $j \in \mathbb{Z}^s$.

Let $L > 1$ be a fixed number, let n, q be two different elements of K_Y, and let us define the mesh function

$$v((l + Yj) \circ h) = (\delta_{ln} - \delta_{lq})\, \chi_{[0, Le]}((l + Yj) \circ h),$$

where $[0, Le] = \{x \in \mathbb{R}^s : 0 \leqslant x \leqslant Le\}$. It can be shown that there exist numbers ξ, ξ' independent of L, h and such that

$$\xi L^{s/p} |e/h|^r \leqslant |v|_{p,r} \leqslant \xi' L^{s/p} |e/h|^r, \qquad 0 \leqslant r \leqslant m.$$

Let us now consider $p^X v$. First, let us observe that $p^Y v$ is nonzero in a neighbourhood of the boundary of $[0, Le]$; strictly speaking, there exists η' such that $p^Y v(x) = 0$ if $x \in [\eta' h, Le - \eta' h]$ or $x \notin [-\eta' h, Le + \eta' h]$. Hence, according to (3.18), there exists η independent of L, h and such that

$p^X v(x)=0$ if $x \in [\eta h, Le-\eta h]$ or $x \notin [-\eta h, Le+\eta h]$. Therefore, there exists a number κ such that

$$|p^X v|_{p,r} \leqslant \kappa L^{(s-1)/p} |e/h|^r |h|^{1/p}, \qquad 0 \leqslant r \leqslant m.$$

Since L can be taken arbitrarily, (2.18) is not satisfied. Thus, the proof is finished. ▮

Proof of the Corollary. It follows from Theorem 3 and condition (2.18) that

$$\|p^X u\|_{p,m} \leqslant \|u\|_{p,m}, \qquad \|p^X u\|_{p,n} \geqslant K \|u\|_{p,n}.$$

Formula (2.19) follows from these estimates and the inequality

$$|u|_{p,n} \leqslant 2^{n-r} |e/h|^{n-r} |u|_{p,r} \qquad \text{if} \quad r \leqslant n \leqslant m.$$

which is true for every $u \in W_p^m(\mathbb{R}_h^s)$ (and follows from the formula $\|\partial^{e_i} u\|_p \leqslant 2\|u\|_p/h_i$). This completes the proof. ▮

3.7. *Proofs of the Results from Section* 2.4

First, let us prove the following formula: if $A \in L_\infty(\mathbb{R}^s)_b$, $f \in L_p(\mathbb{R}^s)$, $u \in L_{p'}(\mathbb{R}_h^s)$, then

$$\langle p^A u, f \rangle = [u, r^A f]. \tag{3.32}$$

It follows from Theorem 1 that if the above assumptions are satisfied, then both sides of formula (3.32) are well defined and

$$\langle p^A u, f \rangle = \int \lim_{n \to \infty} g_n(x)\, dx, \qquad [u, r^A f] = \lim_{n \to \infty} \int g_n(x)\, dx,$$

where

$$g_n(x) = \sum_{l \in \mathbb{Z}^s, |l| \leqslant n} f(x)\, A(x/h - l)\, u(lh).$$

Since the support of A is bounded, $g_n(x) \to f(x)\, p^A u(x)$ almost everywhere. Moreover, g_n are uniformly bounded by an integrable function: $|g_n(x)| \leqslant |f(x)|\, p^A(|u|)(x)$. Hence, the limit of integrals equals the integral of limits and thus, formula (3.32) is proved. ▮

Proof of Lemma 4. According to Theorem 4, there exists a function $A \in \mathscr{P}_l$ such that $r^A p^Y u = u$ for each $u \in m(\mathbb{R}_h^2)$. Thus, it follows from definition (2.26) that

$$rp^Y u = r^A p^Y u = u \qquad \text{for each} \quad u \in X_h. \tag{3.33}$$

Therefore, if $q = p^Y$, then the first condition from (2.24) is satisfied. Next, it follows from the definition of F that

$$\| pu - \psi qu \|_F = \| p^X u - p^Y u \|_2.$$

According to (3.33), every function $u \in X_h$ can be represented as $u = r^A f$, where $f = p^Y u \in W_2^1(\mathbb{R}^2)$, and $\|f\|_{1,2} \leqslant K \|u\|_{1,2}$ due to Theorem 1. Applying Theorem 2, we obtain the estimate

$$\| p^X r^A f - p^Y r^A f \|_2 \leqslant \| p^X r^A f - f \|_2 + \| p^Y r^A f - f \|_2 \leqslant K \|u\|_{1,2} |h|.$$

Thus, $\|pu - \psi qu\|_F \leqslant K \|u\|_{1,2} |h|$, and condition (2.24) is satisfied. ∎

Proof of Formula (2.28). Following (2.21), (2.27), and (2.25),

$$a_h(u, v) = \sum_{i,j=1}^{2} \langle b_{ij} D_i p^Y u, D_j p^Y v \rangle + \langle b p^X u, p^X v \rangle.$$

Applying formula (2.10) we obtain $D_i p^Y u = p^{Y(i)} \partial_i^- u$, and thus, formula (2.28) can be obtained by the use of (3.32) and (3.14). ∎

Acknowledgments

The author is grateful to Professor Zbigniew Ciesielski and Dr. Andrzej Wakulicz for helpful discussions and advice concerning this paper.

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